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Cartwright's Theorem for Harmonic Functions

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Cartwright's theorem [1, p. 180] states that an entire function of exponential type less than π must be bounded on the whole real axis if it is bounded at the integers. In this paper we shall prove that Cartwright's theorem extends to real-valued entire harmonic functions.

THEOREM 1. *Let $u(z)$ be a real-valued entire harmonic function of exponential type less than π. If $|u(n)|$ is bounded for $n = 0, \pm 1, \pm 2, \ldots$, then $|u(x)|$ is bounded for all real x.*

Proof. Let us first construct the entire function $f(z)$ whose real part is $u(z)$, by letting $v(z)$ denote the harmonic conjugate to $u(z)$. Then $f(z) = u(z) + iv(z)$ is entire. To show that $f(z)$ is of exponential type less than π, let $Q(r) = \max_{|z|=r} |u(z)|$. Then by Carathéodory's inequality [1, p. 2] we have that

$$|f(z)| \leqslant |f(0)| + \frac{2r}{R-r}\{Q(R) - u(0)\}, \qquad 0 < r < R,$$

and by taking $R = r + 1$, it follows that $f(z)$ has the same exponential type as $u(z)$.

Now let $g(z) = f(z) + \overline{f(\bar{z})}$. Then $g(z)$ is an entire function of exponential type less than π, and for real values of x,

$$g(x) = 2u(x).$$

Consequently, $g(n) = 2u(n)$ is bounded for $n = 0, \pm 1, \pm 2, \ldots$, and by Cartwright's theorem $g(x)$ is bounded for all real x. Hence $u(x)$ is bounded for all x.

An interesting application of Theorem 1 is to entire functions of zero exponential type. It is already known that an entire function of zero exponential type must reduce to a constant if it is bounded at the integers

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(see [1, p. 183] or [2]). However, will such a function reduce to a constant if its real part is bounded at the integers? The example $f(z) = -(1/2)\, iz^2$ shows that this condition is not sufficient. In this case, $u(x, y) = xy$ is bounded at the integers. In fact, it vanishes there, yet $f(z)$ is not constant. However, the possibility does exist that an entire function of zero exponential type reduces to a constant if its real part is bounded along two parallel lines of lattice points, and this is precisely what we shall prove.

THEOREM 2. *Let $f(z) = u(z) + iv(z)$ be an entire function of zero exponential type. If $|u(n)| = O(1)$ and $|u(n+i)| = O(1)$, $n = 0, \pm 1, \pm 2, \ldots$, then $f(z)$ is a constant.*

Proof. Let $g(z) = f(z) + \overline{f(\bar{z})}$. Then $g(z)$ is an entire function of zero exponential type, and

$$g(x) = 2u(x).$$

In particular,

$$g(n) = 2u(n), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Since $u(z)$ is bounded at the integers, it is bounded on the whole real axis, by Theorem 1. Therefore $g(z)$ is an entire function of zero exponential type that is bounded on the real axis and as such must be constant, so that

$$g(z) = f(z) + \overline{f(\bar{z})} = K_1.$$

In particular, for $z = z + i$ we have

$$f(z+i) = -\overline{f(\bar{z} - i)} + K_1. \tag{1}$$

Now let $h(z) = f(z+i) + \overline{f(\bar{z} + i)}$. Since

$$h(n) = 2u(n+i)$$

we conclude as above that $h(z)$ is a constant. Therefore

$$f(z+i) = -\overline{f(\bar{z} + i)} + K_2. \tag{2}$$

Equations (1) and (2) give

$$f(z+i) - f(z-i) = K_3, \qquad K_3 = K_2 - K_1. \tag{3}$$

Differentiate Eq. (3) to obtain

$$f'(z+i) = f'(z-i).$$

That is, $f'(z)$ is an entire function of exponential type with period $2i$. Such a function has the form

$$f'(z) = \sum_{K=-n}^{n} a_K e^{K\pi z}.$$

Since $f'(z)$ has zero exponential type, $n=0$, and $f'(z)$ is a constant. If this constant is nonzero, then $f(z) = u(z) + iv(z)$ must be a linear polynomial, contradicting the assumption that $u(z)$ is bounded at the integers. Therefore $f'(z) = 0$ and $f(z)$ is a constant.

References

1. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.
2. R. Brück, An extension of Carlson's theorem for entire functions of exponential type, *J. Math. Anal Appl.* **147** (1990), 372–374.